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BRANCHING IN THE ENUMERATION DEGREES OF THE Σ_2^0 SETS

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ABSTRACT

We show that every incomplete Σ_2^0 enumeration degree is meet-reducible in the structure of the enumeration degrees of the Σ_2^0 sets.

1. Introduction

Informally, a set A is enumeration reducible to a set B if there is an effective procedure for enumerating A, given *any* enumeration of B. Following [FR59] and [Rog67], this is usually formalized using the notion of an enumeration operator.

Definition 1.1: A mapping $\Phi\colon 2^\omega \longrightarrow 2^\omega$ is an **enumeration operator** (or, simply an *e*-**operator**), if there exists a computably enumerable set W such that, for each set B,

$$\Phi^B = \{x|\ (\exists u)[\langle x, u\rangle \in W \,\&\, D_u \subseteq B]\},$$

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where D_u denotes the finite set with canonical index u, and $\langle \ , \ \rangle$ denotes the usual pairing function.

If the c.e. set W_z defines the e-operator Φ in the sense of the above definition, then we let $\Phi = \Phi_z$. Let $\{W_z | \ z \in \omega\}$ be the standard enumeration of the c.e. sets: we get a corresponding enumeration $\{\Phi_z | \ z \in \omega\}$ of the e-operators. If $\{W_z^s | \ s \in \omega\}$ is a computable enumeration of W_z (in the sense of [Soa87, p. 34]), then we get a corresponding **computable enumeration** $\{\Phi_z^s | \ s \in \omega\}$ of the e-operator Φ_z. We will refer in the following to some fixed computable sequence $\{W_{z,s} | \ z, s \in \omega\}$ of finite sets, such that, for every z, $\{W_z^s | \ s \in \omega\}$ is a computable enumeration of W_z.

Given sets A, B, we say that A is **enumeration reducible** (or, simply, e-**reducible) to** B (notation: $A \leq_e B$), if there exists some e-operator Φ such that $A = \Phi^B$.

It is easily seen that $\leq_e$ is a preordering relation. Let $\equiv_e$ denote the equivalence relation generated by $\leq_e$. The $\equiv_e$-equivalence class of a set A (denoted by $\deg_e(A)$) is called the **enumeration degree** (or, simply, the e-**degree**) of A. On e-degrees the reducibility $\leq_e$ originates a partial ordering relation (denoted by $\leq$). We therefore get a degree structure $\langle \mathfrak{D}_e, \leq \rangle$, where $\mathfrak{D}_e$ is the collection of all e-degrees and $\leq$ is defined by: $[A]_e \leq [B]_e$ if and only if $A \leq_e B$. In fact $\mathfrak{D}_e$ is an upper semilattice with least element $\mathbf{0}_e$ and binary operation $\cup$: the least element $\mathbf{0}_e$ is the e-degree of the c.e. sets, and $[A]_e \cup [B]_e = [A \oplus B]_e$, with $A \oplus B = \{2x | \ x \in A\} \bigcup \{2x+1 | \ x \in B\}$. The reader may consult [Coo90] and [Sor97] for an extensive survey and bibliography on the e-degrees.

An important class of e-degrees is constituted by the Σ_2^0 e-degrees, i.e. the e-degrees of the Σ_2^0 sets. It is known, see [Coo84] and [McE85], that the Σ_2^0 e-degrees coincide with the structure $\mathfrak{G} = \mathfrak{D}_e(\leq \mathbf{0}_e')$, where $\mathbf{0}_e' = \deg_e(\overline{K})$, $\overline{K}$ being the complement of the halting set K: in fact, $A \leq_e \overline{K}$ if and only if $A \in \Sigma_2^0$.

Although, under several respects, $\mathfrak{G}$ can be viewed as the e-degree theoretic analog of the structure $\mathfrak{R}$ of the Turing degrees of the c.e. sets (as suggested for instance by Cooper: see the density theorem for $\mathfrak{G}$, [Coo84]; see also [LS92]), there are striking elementary differences between the two structures. For instance, [Ahm91] shows (in contrast with the Lachlan Nondiamond Theorem for $\mathfrak{R}$, see [Soa87, p. 162]) that there exist (in fact, low) e-degrees $\mathbf{a}, \mathbf{b} \in \mathfrak{G}$ such that $\mathbf{a} \cup \mathbf{b} = \mathbf{0}_e'$ and $\mathbf{a} \cap \mathbf{b} = \mathbf{0}_e$.

We show in this paper another elementary difference between $\mathfrak{G}$ and $\mathfrak{R}$, that relates to the notion of branching.

Definition 1.2: Let $\mathfrak{P} = \langle \mathfrak{P}, \leq \rangle$ be a partial order. We say that an element $c \in P$ is **branching** (or **meet-reducible**) if

$$(\exists a \in P)(\exists b \in P)[c < a \;\&\; c < b \;\&\; c = a \wedge b].$$

An element $c \in P$ is called **nonbranching** if it is not branching.

[Lac66] proved the existence of incomplete nonbranching elements in $\mathfrak{R}$. Subsequently, [Fej83] proved that the nonbranching elements of $\mathfrak{R}$ are dense. [Sla91] proved the density of the branching elements of $\mathfrak{R}$.

We prove here a rather surprising result for $\mathfrak{S}$. All elements $\mathbf{a} \in \mathfrak{S}$, with $\mathbf{a} < \mathbf{0}'_e$, are branching in $\mathfrak{S}$:

THEOREM 1.3: *For every incomplete Σ_2^0 enumeration degree* $\mathbf{c}$ *there exist enumeration degrees* $\mathbf{a}, \mathbf{b} \leq \mathbf{0}'_e$ *such that:*

(1) $\mathbf{a} \cup \mathbf{c} \not\leq \mathbf{c}$, $\mathbf{b} \cup \mathbf{c} \not\leq \mathbf{c}$,

(2) $\mathbf{c} = (\mathbf{a} \cup \mathbf{c}) \cap (\mathbf{b} \cup \mathbf{c})$.

In the following, suppose that L is a Σ_2^0 set such that $L <_e \overline{K}$, and suppose we are given some Σ_2^0 approximation $\{L^s \mid s \in \omega\}$ to L, i.e. a computable sequence of finite sets such that

$$L = \{x \mid (\exists t)(\forall s \geq t)[x \in L^s]\}.$$

For more on Σ_2^0-approximations, see [LS92]. Finally, let

$$\overline{K}^s = \{x < s \mid x \notin K^s\}$$

(where $\{K^s \mid s \in \omega\}$ is a computable approximation to the halting set K).

2. The requirements

Given $L <_e \overline{K}$, we will construct Σ_2^0 sets A, B by stages. At stage s of the construction, given any expression $\mathcal{A}$, we will often write $\mathcal{A}[s]$ to denote the evaluation of the expression at stage s: see [Soa87, p. 315] for this notation.

If, at stage s, we define the current value of a set $X[s]$, we will write $x \searrow X[s]$ to mean that we enumerate x (or x gets enumerated) into $X[s]$ (hence $x \in X[s]$) and $x \nearrow X[s]$ to mean that we extract x (or x gets extracted) from $X[s]$ (hence $x \notin X[s]$). If E is a finite set, we use similar notations: $E \searrow X[s]$ (i.e. $x \searrow X[s]$, all $x \in E$) and $E \nearrow X[s]$ (i.e. $x \nearrow X[s]$, all $x \in E$).

Let $\{(\Phi_i, \Psi_i)\}_{i\in\omega}$ be an effective listing of all pairs of e-operators.

In order to prove Theorem 1.3, we want to construct Σ^0_2 sets A, B satisfying the following requirements, for every $i, k \in \omega$:

$$\begin{array}{ll} \mathcal{P}_i: & Z_i = \Phi_i^{A\oplus L} = \Psi_i^{B\oplus L} \Rightarrow Z_i = \Gamma_i^L, \\ \mathcal{N}_k^A: & A = \Phi_k^L \Rightarrow \overline{K} = \Delta^L_{A,k}, \\ \mathcal{N}_k^B: & B = \Phi_k^L \Rightarrow \overline{K} = \Delta^L_{B,k}, \end{array}$$

where $\Gamma_i, \Delta_{A,k}, \Delta_{B,k}$ are e-operators to be constructed.

We say that a requirement $\mathcal{R}$ is a $\mathcal{P}$-requirement if, for some i, $\mathcal{R} = \mathcal{P}_i$; in a similar way, we talk about $\mathcal{N}$-requirements, $\mathcal{N}^A$-requirements and $\mathcal{N}^B$-requirements.

We order the requirements according to the following linear ordering (called the **priority** ordering of the requirements):

$$\cdots \mathcal{P}_i < \mathcal{N}_i^A < \mathcal{N}_i^B < \mathcal{P}_{i+1} < \mathcal{N}_{i+1}^A < \mathcal{N}_{i+1}^B \cdots$$

with $i \in \omega$.

3. The strategies

We outline the strategies used, in isolation, to meet the requirements.

3.1 THE ATOMIC MODULES.

3.1.1 The requirement $\mathcal{P}_i$. For simplicity, let us drop the subscript i; let $Z = \Phi^{A\oplus L} \cap \Psi^{B\oplus L}$.

If $x \in Z - \Gamma^L$, then choose finite sets $\alpha, \lambda^A, \beta, \lambda^B$ such that

$$\langle x, \alpha\oplus\lambda^A\rangle \in \Phi \quad \langle x, \beta\oplus\lambda^B\rangle \in \Psi$$

and

$$\alpha\oplus\lambda^A \subseteq A\oplus L \quad \beta\oplus\lambda^B \subseteq B\oplus L,$$

and enumerate an axiom $\langle x, \lambda\rangle \in \Gamma$, with $\lambda \supseteq \lambda^A \cup \lambda^B$.

3.1.2 The requirement $\mathcal{N}_k^A$. For simplicity let us drop the indices A, k.

If all numbers $y < x$ have been chosen, and no such y is currently at 2(a), or 3(i) of the basic module below, then choose x; pick up a new number c_x:

1. if $\overline{K}(x) = \Delta^L(x)$, then do nothing;
2. if $x \in \overline{K} - \Delta^L$, then define $c_x \in A$:

(a) wait for $c_x \searrow \Phi^L$;

(b) choose an axiom $\langle c_x, \lambda\rangle \in \Phi$ such that $\lambda \subseteq L$ and enumerate the axiom $\langle x, \lambda\rangle \in \Delta$;

(c) return to 1.;

3. if $x \in \Delta^L - \overline{K}$ then extract c_x from A;

(i) wait for $c_x \nearrow \Phi^L$;

(ii) return to 1.

The numbers c_x will be called **coding markers**.

3.1.3 The requirement $\mathcal{N}_k^B$. The module in this case is of course similar to the module for $\mathcal{N}_k^A$, but replacing A with B and $\Delta_{A,k}$ with $\Delta_{B,k}$. We skip the obvious details.

3.2 ANALYSIS OF OUTCOMES. We briefly discuss the possible outcomes of the above strategies.

3.2.1 The requirement $\mathcal{P}_i$. For simplicity, let us drop the subscript i.

The strategy here aims at defining suitable axioms $\langle x, \lambda\rangle \in \Gamma$, for $x \in Z$. Note that if, eventually, $\alpha \subseteq A$ and $\beta \subseteq B$ (i.e. the extracting activity of the $\mathcal{N}$-requirements does not interfere with $\mathcal{P}$, see Section 3.3), then $x \nearrow Z \Rightarrow z \nearrow \Gamma^L$ by automatic Γ-rectification: indeed, if $\alpha \subseteq A$ and $\beta \subseteq B$ and $x \notin Z$, then $\lambda^A \cup \lambda^B \not\subseteq L$, hence $\lambda \not\subseteq L$ (where λ^A, λ^B are as in 3.1.1 and $\lambda \supseteq \lambda^A \cup \lambda^B$).

3.2.2 The requirement $\mathcal{N}_k^A$. For simplicity, let us drop the indices A and k.

The finitary outcome 2(a) corresponds to $c_x \in A - \Phi^L$; infinitely many loops through 2(c), in relation to some x, correspond to the case $c_x \in A - \Phi^L$. The finitary outcome 3(i) corresponds to $c_x \in \Phi^L - A$. Infinitely many loops through 3(ii) imply $x \notin \Delta^L$. If no x gets stuck at 2(a) or 3(i), and yields infinitely many loops through 2(c), then we get $\overline{K} = \Delta^L$ (contradicting that $L <_e \overline{K}$). Since $L <_e \overline{K}$, there must exist some number x such that the only allowed outcomes are therefore the finitary outcomes 2(a), 3(i) or the infinitary outcome 2(c).

3.2.3 The requirement $\mathcal{N}_k^B$. See the discussion relative to the outcomes of $\mathcal{N}_k^A$, but replacing A with B and $\Delta_{A,k}$ with $\Delta_{B,k}$.

3.3 INTERACTIONS BETWEEN REQUIREMENTS. The extracting activity of the $\mathcal{N}$-requirements conflicts with the activity of the $\mathcal{P}$-requirements, consisting in defining Γ-axioms. We explain below the nature of these conflicts and how to combine the different strategies.

3.3.1 A $\mathcal{P}$-requirement below an $\mathcal{N}$-requirement. We consider only the case of an $\mathcal{N}$-requirement of the form $\mathcal{N}_k^A$, above some $\mathcal{P}_i$: the case of a requirement of the form $\mathcal{N}_k^B$ is similar.

For simplicity, let us omit the subscripts k and i, and the superscript A.

A somewhat problematic case is when we go through 3. of the basic module of $\mathcal{N}$ on behalf of infinitely many numbers x, ending up with a (possibly) infinite set V (consisting of numbers c_x corresponding to numbers x such that $x \notin \overline{K}$) being extracted from A. How does $\mathcal{P}$, acting after $\mathcal{N}$, account for this infinitary extracting activity?

We measure the length of agreement between $\overline{K}$ and Δ^L by a length of agreement function $\ell(k,s)$, such that $\overline{K} = \Delta^L \Leftrightarrow \lim_s \ell(k,s) = +\infty$: thus $\ell = \liminf_s \ell(k,s)$ exists and is finite.

We will guarantee that

$$A = \Phi^L \Rightarrow \overline{K} = \Delta^L,$$

so that, eventually, $A \neq \Phi^L$.

The number ℓ will be of the form $\ell = \langle x, u \rangle$, with $\overline{K}(x) \neq \Delta^L(x)$. We can therefore distinguish two outcomes: a finitary one ($x \in \Delta^L - \overline{K}$), and an infinitary one ($x \in \overline{K} - \Delta^L$; this latter outcome may be infinitary since there might exist infinitely many stages s such that, at stage s, $x \in \Delta^L$). The problem here is that while working below outcome ℓ, $\mathcal{P}$ cannot foresee which numbers y, that are currently in $\overline{K}$, will be later removed from $\overline{K}$, forcing the strategy to extract the corresponding number c_y from A. We go around this problem by extracting all current c_y's, corresponding to actions undertaken to the right of the current path (i.e. with $y > \ell$), pending our decision to enumerate again in A a new c_y at the next stage s at which $y \leq \ell(k,s)$ and $y \in \overline{K}^s$ (so that eventually, for some c_y, we have that $c_y \in A$ if $y \in \overline{K}$ and $\lim \ell(k,s) = +\infty$).

We hence arrange things so that the extracting activity of $\mathcal{N}$ results in extracting a (possibly infinite) computable set V from A, by simply amending the basic module with the addition of the following clause:

if $y > \ell$, then extract also c_y from A.

The $\mathcal{P}$-activity (consisting in enumerating Γ-axioms) can easily deal with these extractions from its position on the tree of outcomes.

3.3.2 An $\mathcal{N}$-requirement below a $\mathcal{P}$-requirement. We consider only the case of an $\mathcal{N}$-requirement of the form $\mathcal{N}_k^A$ below some $\mathcal{P}_i$, the case of $\mathcal{N}_k^B$ being similar, and we omit obvious subscripts and superscripts.

It is clear from the discussion in the previous subsection that the extracting activity of $\mathcal{N}$ interferes with the strategy of $\mathcal{P}$ in that $\mathcal{N}$-extractions may result in forcing numbers z to leave $\Phi^{A\oplus L}$; thus $z \nearrow Z$, and this implies that we need $z \nearrow \Gamma^L$, if we hope to maintain the equation $Z = \Gamma^L$.

We need therefore to consider what happens when, for some z, at some stage, $z \in Z = \Phi^{A\oplus L} \cap \Psi^{B\oplus L}$, and we consequently enumerate an axiom $\langle z, \lambda\rangle \in \Gamma$, with $\lambda \subseteq L$, in order to have $z \in \Gamma^L$, and subsequent extracting activity demanded by $\mathcal{N}$ will force z to leave $\Phi^{A\oplus L}$ and thus, Z, so that we need to extract z from Γ^L.

Enumerating Γ-axioms at sufficiently large stages. We deal with the difficulties entailed by the Π_3^0 hypothesis $\Phi^{A\oplus L} = \Psi^{B\oplus L}$ of $\mathcal{P}$, by dispersing down through the tree of outcomes our attempts at diagonalizing against $\Phi^{A\oplus L} = \Psi^{B\oplus L}$, as well as the enumeration of the Γ-axioms. Therefore for almost all x, axioms of the form $\langle x, \lambda\rangle \in \Gamma$ will be enumerated only after acting at $\mathcal{N}$. Thus, for almost all x, we enumerate axioms $\langle x, \lambda\rangle \in \Gamma$ only at stages s such that $\ell(k,s) \geq \ell$, where $\ell = \langle x, u\rangle = \liminf_s \ell(k,s)$. Clearly, enumeration of a Γ-axiom below ℓ, if ℓ is the finitary outcome at $\mathcal{N}$, does not present any problem. On the other hand, enumeration of Γ-axioms below or to the right of the infinitary outcome at $\mathcal{N}$ (that is, $x \in \overline{K} - \Delta^L$) can also be easily dealt with for the following reason: if we work at stages s such that $\ell(k,s) > \langle x, u\rangle$, then we are assuming that $x \in \Delta^L[s]$, thus we enumerate axioms $\langle z, \lambda\rangle \in \Gamma$, with λ containing the Δ^L-use of x (i.e. $x \in \Delta^\lambda$ and $\lambda \subseteq L$ at stage s). Since $x \notin \Delta^L$, it follows that $\lambda \not\subseteq L$, thus none of these axioms applies to get $z \in \Gamma^L$.

We may therefore conclude that there are only finitely many numbers x such that (letting V be the possibly infinite set eventually extracted by $\mathcal{N}$) we have that $x \nearrow Z$, due to $V \nearrow A$, but $x \in \Gamma^L$. These numbers are included among those numbers x for which we enumerate axioms $\langle x, \lambda\rangle \in \Gamma$ before acting at $\mathcal{N}$.

We deal with these finitely many numbers x by looking for opportunities of diagonalization, thus getting $\Phi^{A\oplus L}(x) \neq \Psi^{B\oplus L}(x)$, as follows: if $x \notin \Phi^{A\oplus L}$, due to the extracting activity of $\mathcal{N}$, then we restrain some finite $\beta \subseteq B$ such that $x \in \Psi^{\beta\oplus L}$. If, following this action, no L-change occurs yielding $x \nearrow \Psi^{\beta\oplus L}$ (and thus $x \nearrow \Gamma^L$), then we win the requirement $\mathcal{P}$, since we get $x \in \Psi^{B\oplus L} - \Phi^{A\oplus L}$; otherwise we get $x \nearrow \Gamma^L$, thus restoring the equation $Z(x) = \Gamma^L(x)$: in this latter case we drop any previous restraint.

Here is a more schematic description of the above strategy.

If $x \in \Gamma^L$ and $x \nearrow Z$ (due to $\mathcal{N}$) and $x \notin \Phi^{A\oplus L}$, then pick finite sets β and λ such that

(i) $x \in \Psi^{\beta\oplus\lambda}$, $\langle x, \lambda\rangle \in \Gamma$ and $\lambda \subseteq L$: enumerate β in B, and restrain $\beta \subseteq B$;

(ii) wait for $x \nearrow \Psi^{\beta\oplus L}$ (thus it is not the case that $x \in \Gamma^L$, via the axiom $\langle x, \lambda\rangle \in \Gamma$, due to some L-change relative to some element of λ);

(iii) drop any restraint.

Infinitely many loops through (ii) yield $x \notin \Gamma^L$; otherwise $x \in \Psi^{B\oplus L} - \Phi^{A\oplus L}$.

This restraining activity (taken care of by $\mathcal{N}$ and located at the appropriate $(\mathbb{N}, \mathbb{P})$ of the tree of outcomes) does not prevent lower priority requirements from being satisfied, since it is finitary, referring only to finitely many numbers x.

A more detailed discussion of how to combine the strategies will be in reference to the tree of outcomes, described in next section.

4. The tree of outcomes

In this section we define the tree of outcomes T, which is going to be a subset of $\omega^{\langle\omega}$. Let $\mathbb{R}$ be the set of all requirements, and let $\mathbb{P}$ and $\mathbb{N}$ denote the sets of $\mathcal{P}$-requirements and $\mathcal{N}$-requirements, respectively. The set $\mathbb{N}$ is partitioned into $\mathbb{N}^A$ and $\mathbb{N}^B$, i.e. the sets of $\mathcal{N}^A$- and $\mathcal{N}^B$-requirements, respectively. Together with the tree of outcomes, we will define also the **requirement assignment** function, i.e. a function $\mathcal{R} : T \longrightarrow \mathbb{R} \cup (\mathbb{N} \times \mathbb{P}) \cup (\mathbb{P} \times \omega)$.

The elements of T will be called **strings** or **nodes**. We will distinguish the **$\mathbb{P}$-nodes**, the **$\mathbb{N}$-nodes** (partitioned into the $\mathbb{N}^A$- and the $\mathbb{N}^B$-nodes), the **$(\mathbb{N}, \mathbb{P})$-nodes** (again, partitioned into the **$(\mathbb{N}^A, \mathbb{P})$-nodes** and **$(\mathbb{N}^B, \mathbb{P})$-nodes**), and the **$\Gamma$-nodes**. If σ is a $\mathbb{P}$-node, then $\mathcal{R}(\sigma)$ is a $\mathcal{P}$-requirement; if σ is an $\mathbb{N}$-node, then $\mathcal{R}(\sigma)$ is an $\mathcal{N}$-requirement; if σ is an $(\mathbb{N}, \mathbb{P})$-node, then $\mathcal{R}(\sigma) \in \mathbb{N} \times \mathbb{P}$, i.e. $\mathcal{R}(\sigma) = (\mathcal{N}, \mathcal{P})$, where $\mathcal{N}$ is an $\mathcal{N}$-requirement and $\mathcal{P}$ is a $\mathcal{P}$-requirement; finally, if σ is a Γ-node, then $\mathcal{R}(\sigma) = (\mathcal{P}, x)$, where $\mathcal{P}$ is a $\mathcal{P}$-requirement and $x \in \omega$. Along any infinite path of T, the assignment of requirements to nodes is according to the priority listing of the requirements. The meaning of the Γ-nodes and of the $(\mathcal{N}, \mathcal{P})$-nodes will be explained in Subsection 4.2.

Before giving the formal details, we give some intuition underlying the definitions of T and $\mathcal{R}$. With the exception of the $\mathbb{P}$-nodes, having only one outcome, each node has countably many outcomes (with order type ω). Along any infinite branch f of the tree, each $\mathbb{P}$-node is followed by a Γ-node; each Γ-node is followed by an $\mathbb{N}^A$-node; if $\sigma \subset f$ is an $\mathbb{N}^A$-node, and $\mathcal{R}(\sigma) = \mathcal{N}^A_k$, then σ is immediately followed by $k+1$ $(\mathbb{N}, \mathbb{P})$-nodes $\tau_0, \ldots, \tau_k$, where $\mathcal{R}(\tau_i) = (\mathcal{N}^A_k, \mathcal{P}_i)$, with $\mathcal{P}_i$ the i-th $\mathcal{P}$-requirement in order of priority; the last such $(\mathbb{N}, \mathbb{P})$-node is followed by an $\mathbb{N}^B$-node; as for the $\mathbb{N}^A$-nodes, if $\sigma \subset f$ is an $\mathbb{N}^B$-node and $\mathcal{R}(\sigma) = \mathcal{N}^B_k$, then σ is immediately followed by $k+1$ $(\mathbb{N},\mathbb{P})$-nodes; the last such $(\mathbb{N},\mathbb{P})$-node is followed by an $\mathbb{P}$-node; finally, if $\sigma \subset f$ is a $\mathbb{P}$-node, then for every $x \in \omega$ there exists exactly one Γ-node τ such that $\sigma \subset \tau \subset f$ and $\mathcal{R}(\tau) = (\mathcal{R}(\sigma), x)$, and for no $\tau' \subset \sigma$ can we have $\mathcal{R}(\tau') = (\mathcal{R}(\sigma), y)$, for any y.

Definition 4.1: T and $\mathcal{R}$ are defined by induction as follows (we assume that $\mathbb{P} \times \omega$ is ordered as follows: let $(\mathcal{P}_i, x) < (\mathcal{P}_{i'}, x')$ if and only if $\langle i, x\rangle < \langle i', x'\rangle$; when referred to requirements, the term "least" below, unless otherwise specified, always refers to the priority ordering of the requirements):

1. $\emptyset \in T$; $\emptyset$ is a $\mathbb{P}$-node; $\mathcal{R}(\emptyset) = \mathcal{P}_0$;
2. if $\sigma \in T$ and σ is a $\mathbb{P}$-node, then $\sigma^\frown 0 \in T$; $\sigma^\frown 0$ is a Γ-node; finally,
$$\mathcal{R}(\sigma^\frown 0) = \text{least}\{(\mathcal{P}, x) \in \mathbb{P} \times \omega |\ (\forall \tau \subseteq \sigma) \\ [\mathcal{R}(\tau) \neq (\mathcal{P}, x)] \,\&\, (\exists \tau \subseteq \sigma)[\mathcal{R}(\tau) = \mathcal{P}]\};$$
3. if $\sigma \in T$ and σ is a Γ-node, then, for every n, $\sigma^\frown n \in T$ and $\sigma^\frown n$ is an $\mathbb{N}$-node; let
$$\mathcal{R}(\sigma^\frown n) = \text{least}\{\mathcal{N} \in \mathbb{N} |\ (\forall \tau \subseteq \sigma)[\mathcal{R}(\tau) \neq \mathcal{N}]\};$$
4. if $\sigma \in T$ and σ is an $\mathbb{N}^A$-node, then, for every $n \in \omega$, $\sigma^\frown n \in T$; $\sigma^\frown n$ is an $(\mathbb{N}^A, \mathbb{P})$-node; $\mathcal{R}(\sigma^\frown n) = (\mathcal{R}(\sigma), \mathcal{P}_0)$.

   Finally, let $o(\sigma^\frown n) = \sigma$ (see Remark 4.2 below);
5. if $\sigma \in T$ and σ is an $\mathbb{N}^B$-node, then, for every $n \in \omega$, $\sigma^\frown n \in T$; $\sigma^\frown n$ is an $(\mathbb{N}^B, \mathbb{P})$-node; $\mathcal{R}(\sigma^\frown n) = (\mathcal{R}(\sigma), \mathcal{P}_0)$.

   Finally, let $o(\sigma^\frown n) = \sigma$;
6. if $\sigma \in T$ and σ is an $(\mathbb{N}^A, \mathbb{P})$-node, and, say, $\mathcal{R}(\sigma) = (\mathcal{N}^\mathcal{A}, \mathcal{P})$ with $\mathcal{P} \leq \mathcal{N}^\mathcal{A}$, and
$$\{\mathcal{R} \in \mathbb{P} |\ \mathcal{P} < \mathcal{R} \leq \mathcal{N}^\mathcal{A}\} \neq \emptyset,$$
   then let $\sigma^\frown n \in T$, for every $n \in \omega$; the nodes $\sigma^\frown n$ are $(\mathbb{N}^A, \mathbb{P})$-nodes; we define $\mathcal{R}(\sigma^\frown n)$ to be $(\mathcal{N}^\mathcal{A}, \mathcal{P}')$, where $\mathcal{P}'$ is the least requirement $\mathcal{R} \in \mathbb{P}$ such that $\mathcal{P} < \mathcal{R} \leq \mathcal{N}^\mathcal{A}$.

   Let $o(\sigma^\frown n) = o(\sigma)$, for every $n \in \omega$;
7. if $\sigma \in T$ and σ is an $(\mathbb{N}^A, \mathbb{P})$-node, and, say, $\mathcal{R}(\sigma) = (\mathcal{N}^\mathcal{A}, \mathcal{P})$ with $\mathcal{P} \leq \mathcal{N}^\mathcal{A}$, and
$$\{\mathcal{R} \in \mathbb{P} |\ \mathcal{P} < \mathcal{R} \leq \mathcal{N}^\mathcal{A}\} = \emptyset,$$

then let $\sigma\hat{}n \in T$, for every $n \in \omega$; the nodes $\sigma\hat{}n$ are $\mathbb{N}^B$-nodes; we define $\mathcal{R}(\sigma\hat{}n)$ to be the least $\mathcal{N}^{\mathcal{B}}$-requirement $\mathcal{R}$ with $\mathcal{R} > \mathcal{N}^{\mathcal{A}}$;

8. if $\sigma \in T$ and σ is an $(\mathbb{N}^B, \mathbb{P})$-node, and, say, $\mathcal{R}(\sigma) = (\mathcal{N}^{\mathcal{B}}, \mathcal{P})$ with $\mathcal{P} \leq \mathcal{N}^{\mathcal{B}}$, and
$$\{\mathcal{R} \in \mathbb{P} | \; \mathcal{P} < \mathcal{R} \leq \mathcal{N}^{\mathcal{B}}\} \neq \emptyset,$$
then let $\sigma\hat{}n \in T$, for every $n \in \omega$; the nodes $\sigma\hat{}n$ are $(\mathbb{N}^B, \mathbb{P})$-nodes; we define $\mathcal{R}(\sigma\hat{}n)$ to be $(\mathcal{N}^{\mathcal{B}}, \mathcal{P}')$, where $\mathcal{P}'$ is the least requirement $\mathcal{R} \in \mathbb{P}$ such that $\mathcal{P} < \mathcal{R} \leq \mathcal{N}^{\mathcal{B}}$.

   Let $o(\sigma\hat{}n) = o(\sigma)$, for every $n \in \omega$;

9. if $\sigma \in T$ and σ is an $(\mathbb{N}^B, \mathbb{P})$-node, and, say, $\mathcal{R}(\sigma) = (\mathcal{N}^{\mathcal{B}}, \mathcal{P})$ with $\mathcal{P} \leq \mathcal{N}^{\mathcal{B}}$, and
$$\{\mathcal{R} \in \mathbb{P} | \; \mathcal{P} < \mathcal{R} \leq \mathcal{N}\} = \emptyset,$$
then let $\sigma\hat{}n \in T$, for every $n \in \omega$; the nodes $\sigma\hat{}n$ are $\mathbb{P}$-nodes; we define $\mathcal{R}(\sigma\hat{}n)$ to be the least $\mathcal{P}$-requirement $\mathcal{P} \geq \mathcal{N}^{\mathcal{B}}$.

We will sometimes write $\sigma \in T^{\mathbb{N}}$, $\sigma \in T^{\mathbb{N}^A}$, $\sigma \in T^{\mathbb{N}^B}$, if σ is an $\mathbb{N}$-, $\mathbb{N}^A$-, or an $\mathbb{N}^B$-node, respectively.

Remark 4.2: We notice:

1. if σ is an $(\mathbb{N}, \mathbb{P})$-node, then $o(\sigma)$ denotes the largest $\mathbb{N}$-node $\tau \subset \sigma$;
2. if $\mathcal{R}(\sigma) = \mathcal{P}_i$, we will sometimes happen to write
$$Z_\sigma = \Phi_\sigma^{A \oplus L} = \Psi_\sigma^{B \oplus L} \Rightarrow Z_\sigma = \Gamma_\sigma^L$$
instead of
$$Z_i = \Phi_i^{A \oplus L} = \Psi_i^{B \oplus L} \Rightarrow Z_i = \Gamma_\sigma^L$$
(and similarly Z_σ for Z_i, Φ_σ for Φ_i, etc.).

   Similarly we may write
$$A = \Phi_\sigma^L \Rightarrow \overline{K} = \Delta_{A,\sigma}^L$$
instead of
$$A = \Phi_k^L \Rightarrow \overline{K} = \Delta_{A,\sigma}^L$$
(and similarly Φ_σ for Φ_k, etc.), if $\mathcal{R}(\sigma)$ is an $\mathcal{N}^{\mathcal{A}}$-requirement; we use similar notations for $\mathbb{N}^B$-nodes.

Definition 4.3: Let $\{\xi_\sigma | \sigma \in T\}$ be a computable partition of ω into infinite computable sets. The elements of ξ_σ may be chosen to be appointed as **σ-coding markers**.

4.1 NOTATION AND TERMINOLOGY FOR STRINGS. We use standard terminology and notations for strings. In particular, given any $\sigma \in T$, let $|\sigma|$ denote the length of σ.

Given $\sigma, \tau \in T$, let $\sigma \preceq \tau$ if and only if either $\sigma \subseteq \tau$, or $y(\sigma,\tau)\downarrow$ and $\sigma(y(\sigma,\tau)) \leq \tau(y(\sigma,\tau))$, where $y(\sigma,\tau)$ is the least number, if defined, such that $y < |\sigma|, |\tau|$ and $\sigma(y) \neq \tau(y)]$. We say that σ is **to the left of** τ (notation: $\sigma \prec_L \tau$), if $\sigma \preceq \tau$, but $\sigma \not\subseteq \tau$. Given a string σ and a number y, the symbol $\sigma \upharpoonright y$ denotes the initial segment of σ having length y. If $|\sigma| > 0$, then let $\sigma^- = \sigma \upharpoonright |\sigma| - 1$.

If $\sigma \in T$, and $n \in \omega$ is such that $\sigma\hat{\ }n \in T$, then we say that n is an **outcome at** σ. Finally, if $\tau \subseteq \sigma$ and $\tau = \tau^-\hat{\ }x$, then we say that x is the **outcome at** τ^- **along** σ.

4.2 ANALYSIS OF TREE OUTCOMES.

We give now intuition for the construction, which is formally explained in the next section.

1. If σ is a $\mathbb{P}$-node (say $\mathcal{R}(\sigma) = \mathcal{P}_i$), then we observe that we have no distinct outcomes at σ. We just regard σ as the node at which we start our strategy for the corresponding $\mathcal{P}$-requirement, by routinely updating the operator Γ_σ. The eventual success of the strategy will need the cooperation of the lower priority $\mathcal{N}$-requirements. The updating strategy will be dispersed through the infinitely many Γ-nodes $\tau \supseteq \sigma$ with $\mathcal{R}(\tau) = (\mathcal{P}_i, x)$, for some x.

2. Let σ be a Γ-node (say $\mathcal{R}(\sigma) = (\mathcal{P}_i, x)$). The Γ-node σ is devoted to defining suitable axioms $\langle x, \lambda\rangle \in \Gamma_\pi$, where $\pi \subseteq \sigma$ is such that $\mathcal{R}(\pi) = \mathcal{P}_i$. If $x \in \Phi_i^{A\oplus L} \cap \Psi_i^{B\oplus L} - \Gamma_\pi^L$ at stage s, then we suitably choose finite sets $\alpha, \beta, \lambda^A, \lambda^B$ such that, at stage s,
$$x \in \Phi_i^{\alpha\oplus\lambda^A} \cap \Psi_i^{\beta\oplus\lambda^B}$$
and $\alpha \oplus \lambda^A \subseteq A \oplus L$, and $\beta \oplus \lambda^B \subseteq B \oplus L$. We enumerate an axiom $\langle x, \lambda\rangle \in \Gamma_\pi$, where $\lambda \subseteq L^s$ and $\lambda \supseteq \lambda^A \cup \lambda^B$, and λ is large enough to contain all finite sets $\lambda(\rho, s)$ such that $\rho \subseteq \sigma$ and $\lambda(\rho, s) \subseteq L^s$. We now briefly explain what the sets $\lambda(\rho, s)$ are, for $\rho \subseteq \sigma$.

- If ν is a Γ-node, and n is the outcome at ν along σ, i.e. $\rho = \nu\hat{\ }n \subseteq \sigma$, then we let
$$\lambda(\rho, s) = \bigcup\{\lambda |\ \lambda \in D_k\}$$
where D_k (the finite set with canonical index k), also denoted by $D(\nu, s)$ in the construction, is the current guess, at stage s, at the eventually finite collection of finite sets λ's such that $\langle x, \lambda\rangle \in \Gamma_\pi$ and $\lambda \subseteq L$.
- If $\nu \subseteq \sigma$ is an $\mathbb{N}$-node, and ℓ is the outcome at ν along σ, i.e. $\rho = \nu\hat{\ }\ell \subseteq \sigma$, then we single out a suitable set of numbers x such that $x \in \Delta_\nu^L$ at stage s, and, for each such x, we will denote by $\lambda(\nu, x, s)$ a suitably chosen finite set such that $\lambda(\nu, x, s) \subseteq L^s$ and $x \in \Delta_\nu^{\lambda(\nu,x,s)}$: if $C(\rho, s)$ the set of all such numbers x, then finally we let
$$\lambda(\rho, s) = \bigcup_{x\in C(\rho,s)} \lambda(\nu, x, s).$$
- Similarly, if ν is an $(\mathbb{N}, \mathbb{P})$-node (where, say, $\pi' \subseteq \sigma$ is the corresponding $\mathbb{P}$-node, i.e. $\mathcal{R}(\nu) = (\mathcal{N}', \mathcal{R}(\pi'))$, for some $\mathcal{N}'$) such that $\nu \subseteq \sigma$, n is the outcome at ν along σ, i.e. $\rho = \nu\hat{\ }n \subseteq \sigma$, then we will denote by $\lambda(\rho, s)$ the finite set
$$\lambda(\rho, s) = \bigcup_{y\in D_h} \lambda(\nu, y, s)$$
where D_h (also denoted by $E(\nu, s)$ in the construction) is the current guess at the (eventually finite) set of elements leaving $Z_{\pi'}$ as a consequence of the extracting activity of $\mathcal{R}(o(\nu))$, but $D_n \subseteq \Gamma_{\pi'}^L$ at stage s; by $\lambda(\nu, y, s)$ we mean some suitably chosen finite set such that $y \in \Gamma_{\pi'}^{\lambda(\nu,y,s)}$ and $\lambda(\nu, y, s) \subseteq L^s$.

Notice that, in all cases, $\lambda(\rho, s) \subseteq L^s$.

Remark 4.4: Notice that any L-change at some later stage t, relative to any of these sets $\lambda(\rho, s)$ (i.e. $\lambda(\rho, s) \not\subseteq L^t$), will entail $\lambda \not\subseteq L^t$, for the λ used in the new Γ_π axiom. This is a crucial point for the success of $\mathcal{R}_\sigma$: if π is on the true path, then the construction guarantees that all axioms $\langle x, \lambda\rangle \in \Gamma_\pi$ defined while acting at a stage s at some string to the right of the true path are such that λ will contain some set $\lambda(\rho, s)$ such that $\lambda(\rho, s) \not\subseteq L$, so these axioms do not apply to get $x \in \Gamma^L$.

If n is the outcome of σ at s, at $\sigma\hat{\ }n$, for each $\lambda \in D_n$ we restrain finite sets $\alpha(\sigma, \lambda, s) \subseteq A$ and $\beta(\sigma, \lambda, s) \subseteq B$ such that

$$x \in \Phi^{\alpha(\sigma,\lambda,s)\oplus\lambda} \cap \Psi^{\beta(\sigma,\lambda,s)\oplus\lambda}$$

(let $\alpha(\sigma, \lambda, s) = \emptyset$ and $\beta(\sigma, \lambda, s) = \emptyset$ if no such finite sets exist) by restraining the finite sets

$$\alpha(\sigma\hat{\ }n, s) = \bigcup_{\lambda \in D_n} \alpha(\sigma, \lambda, s), \qquad \beta(\sigma\hat{\ }n, s) = \bigcup_{\lambda \in D_n} \beta(\sigma, \lambda, s)$$

in A and B, respectively. Following this restraining action, if $\lambda \subseteq L$ for some $\lambda \in D_n$, then the only $\mathcal{N}$-requirements that are allowed to force $x \nearrow Z_i$ are those of higher priority than $\mathcal{R}_\sigma$.

Notice that we drop any restraint when we move along $\sigma\hat{\ }0$: the tree outcome 0 corresponds to the case $x \notin \Gamma^L_\pi$.

3. Let σ be an $\mathbb{N}$-node. Assume for simplicity that σ is an $\mathbb{N}^A$-node, the case of an $\mathbb{N}^B$-node being similar. We define a length of agreement function $\ell(\sigma, s)$, and we show (with σ on the true path),

$$\lim_s \ell(\sigma, s) = +\infty \Rightarrow \overline{K} = \Delta^L_\sigma.$$

On the other hand, the construction guarantees that

$$A = \Phi^L_\sigma \Rightarrow \overline{K} = \Delta^L_\sigma.$$

It then follows that $\liminf_s \ell(\sigma, s) = \ell$ is finite, and thus $A \neq \Phi^L_\sigma$. We give outcome $\ell(\sigma, s)$ at σ at s.

The outcome $\ell(\sigma, s)$ will be of the form $\ell = \langle x, u\rangle$: we aim at getting either $x \in \overline{K} - \Delta^L_\sigma$ (and in this case, for every $s \geq u$, $x \in \overline{K}^s$), or $x \in \Delta^L_\sigma - \overline{K}$ (and in this case, for every $s \geq u$, $x \in \Delta^L_\sigma$ at stage s).

Each coding marker c (see Definition 4.3) will be chosen from ξ_σ: the coding marker of z, when chosen at some stage s, will be denoted by $c(\sigma, z, s)$.

If $\ell = \langle x, u\rangle$ is the outcome at σ at stage s, then we extract from A^s a finite set $V(\sigma, s)$, consisting (modulo higher priority constraints) of all previously appointed coding markers c_y, corresponding to the numbers y such that $y > \ell$, or $y \leq \ell$ and $y \notin \overline{K}^s$.

We refer the reader to Subsection 3.3.1 for a discussion relative to this extracting activity. Notice however that, in accordance with Subsection 3.3.1, if $\ell = \liminf_s \ell(\sigma, s)$ then we do not have $\overline{K}(\ell) \neq \Delta^L_\sigma(\ell)$, but rather $\overline{K}(x) \neq \Delta^L_\sigma(x)$, where $\ell = \langle x, u \rangle$, for some u.

4. Let σ be an $(\mathbb{N}, \mathbb{P})$-node. Assume for simplicity that σ is an $(\mathbb{N}^A, \mathbb{P})$-node, the case of an $(\mathbb{N}^B, \mathbb{P})$-node being similar.

   Let $\mathcal{R}(\sigma) = (\mathcal{N}^A_k, \mathcal{P}_i)$, and let $\pi \subseteq \sigma$ be such that $\mathcal{R}(\pi) = \mathcal{P}_i$. At this node σ we monitor the effects on $\mathcal{R}(\pi)$ of the extracting activity done on behalf of $\mathcal{R}(o(\sigma))$, with $\mathcal{N}^A_k$ of lower priority than $\mathcal{P}_i$ (recall that $\mathcal{R}(o(\sigma)) = \mathcal{N}^A_k$: let us write ν instead of $o(\sigma)$). Suppose that at stage s we need to extract $V(\nu, s)$ from A, as demanded by the strategy for $\mathcal{R}(\nu)$ (let us write $V = V(\nu, s)$). Let us use the symbol E to denote the finite set $E(\sigma, s)$ of numbers such that, at step s:

$$V \nearrow A \Rightarrow E \nearrow Z_i \,\&\, E \subseteq \Gamma^L_\pi$$

   (where, of course, for any $x \in E$, axioms of the form $\langle x, \lambda \rangle \in \Gamma_\pi$ have been previously defined).

   We give outcome $h = h(\sigma, s)$ at σ at s, where h is the canonical index of E. If $E \neq \emptyset$, then we restrain at $\sigma^\frown h$ some finite set $F \subseteq B$ such that $E \subseteq \Psi_i^{F \oplus L}$. We use the symbol $\beta(\sigma^\frown h, s)$ ($\alpha(\sigma^\frown h, s)$, respectively, if σ is an $(\mathbb{N}^B, \mathbb{P})$-node) to denote such a finite set F: in fact, for every $x \in E$, we suitably choose finite sets $\beta(\sigma, x, s)$ (respectively, $\alpha(\sigma, x, s)$ if σ is an $(\mathbb{N}^B, \mathbb{P})$-node) and $\lambda(\sigma, x, s)$ such that $x \in \Psi^{\beta(\sigma,x,s) \oplus \lambda(\sigma,x,s)}$, $\beta(\sigma, x, s) \oplus \lambda(\sigma, x, s) \subseteq B \oplus L[s]$ and $\langle x, \lambda(\sigma, x, s) \rangle \in \Gamma^s_\pi$, and we restrain $\beta(\sigma, x, s) \subseteq B$ ($\alpha(\sigma, x, s) \subseteq A$, respectively, if σ is an $(\mathbb{N}^B, \mathbb{P})$-node), by restraining the finite set

$$\beta(\sigma^\frown h, s) = \bigcup_{x \in E(\sigma,s)} \beta(\sigma, x, s)$$

   in B (respectively, $\alpha(\sigma^\frown h, s) = \bigcup_{x \in E(\sigma,s)} \alpha(\sigma, x, s)$ in A, if σ is an $(\mathbb{N}^B, \mathbb{P})$-node).

   If σ is on the true path, we will show that $h = \liminf_s h(\sigma, s)$ exists. There are two possibilities:

   - If we get outcome 0 at σ infinitely often, then there is no damage caused to $\mathcal{R}(\pi)$ by the extracting activity done on behalf of $\mathcal{R}(\nu)$, since, for all possible x such that $x \nearrow Z_i$ due to $\mathcal{R}(\nu)$-extractions, we get $x \nearrow \Gamma^L_\pi$ due to infinitely many corresponding L-changes.

- Otherwise $D_h \neq \emptyset$. Then, *either* for some $x \in D_h$ our restraining activity at $\sigma^\frown h$ gives $x \in \Psi_\pi^{B\oplus L} - \Phi_\pi^{A\oplus L}$: this yields an outright win of $\mathcal{R}(\pi)$; *or* $x \in \Phi_\pi^{A\oplus L} \cap \Psi_\pi^{B\oplus L}$, for all $x \in D_h$, showing that the $\mathcal{R}(\nu)$-extractions do not interfere with the equation $Z_i = \Gamma_\pi^L$.

  Thus the outcome $D_h \neq \emptyset$ entails a successful diagonalization against the hypothesis $\Phi_\pi^{A\oplus L} = \Psi_\pi^{B\oplus L}$ of the $\mathcal{P}$-requirement corresponding to the $(\mathbb{N}, \mathbb{P})$-node σ.

5. The construction

The construction is by stages and aims to define suitable recursive sequences of finite sets $\{A^s |\ s \in \omega\}$ and $\{B^s |\ s \in \omega\}$, such that the Σ_2^0 sets

$$A = \{x |\ (\exists t)(\forall s \geq t)[x \in A^s]\} \quad \text{and} \quad B = \{x |\ (\exists t)(\forall s \geq t)[x \in B^s]\}$$

satisfy the requirements of Section 2.

At stage s we define a string $\delta_s \in T$ (with $|\delta_s| = s$), together with the values of several parameters. The intuitive meaning of all relevant parameters has been already explained in the previous section.

For every $\sigma \in T$ and stage s, let

$$t(\sigma, s) = \begin{cases} \max\{t < s |\ \sigma \subseteq \delta_t\} & \text{if any,} \\ s & \text{otherwise.} \end{cases}$$

Definition 5.1: Throughout the following, given any e-operator Φ and any Σ_2^0 set X with a Σ_2^0-approximation $\{X^s\}_{s\in\omega}$, by $\{\Phi_s^{X^s}\}_{s\in\omega}$ we mean the Σ_2^0-approximation to Φ^X defined in [MC85, Proposition 5]. While acting at σ at stage s (i.e. $\sigma \subseteq \delta_s$), given any Σ_2^0 set X with a given Σ_2^0-approximation $\{X^s\}_{s\in\omega}$, we will write (for v such that $t(\sigma, s) \leq v \leq s$): $x \in X[v]$, if

$$(\forall u)[t(\sigma, s) \leq u \leq v \Rightarrow x \in X^u].$$

Definition 5.2: Let $P(y, s)$ be any relation. If $P(y, s)$ holds, then let

$$t_y(s) = \text{least } \{t |\ P(y, t) \,\&\, (\forall u)[t \leq u \leq s \Rightarrow P(y, u)]\}.$$

We say that we **optimally choose** y **for** P at stage s if y is the least number among those with minimal $t_y(s)$ (in fact, y can be (the code of) a finite set or a pair of finite sets, etc.).

At step s, any parameter p retains the same value as at the preceding stage, unless otherwise specified by the construction. Any parameter p is by default undefined (i.e. $p =\uparrow$ if p ranges through the numbers, and $p = \emptyset$, if p ranges through the finite sets).

The e-operators $\Gamma_\sigma, \Delta_{A,\sigma}, \Delta_{B,\sigma}$ will be defined through computable approximations (modulo identification of each e-operator with the corresponding c.e. set): at stage s we define $\Gamma^s_\sigma, \Delta^s_\sigma, \Delta^s_\sigma$.

5.1 STEP 0. Let $\delta_0 = \emptyset$. For every $\sigma \in T$ let

$$V(\sigma,0) = E(\sigma,0) = \alpha(\sigma,0) = \beta(\sigma,0) = \lambda(\sigma,0) = \lambda^A(\sigma,0) = \lambda^B(\sigma,0) = \emptyset.$$

For every $\sigma \in T$ and $z \in \omega$, let

$$\alpha(\sigma,z,0) = \beta(\sigma,z,0) = \lambda(\sigma,z,0) = \emptyset;$$

let $c(\sigma,z,0) =\uparrow$, $\ell(\sigma,0) = h(\sigma,0) =\uparrow$, $\Gamma^0_\sigma = \Delta^0_\sigma = \emptyset$.

Finally, let $A^0 = \emptyset$ and $B^0 = \emptyset$.

5.2 STEP $s+1$. Assume that we have already defined $\delta_{s+1}\restriction n$ (with $\delta_{s+1}\restriction 0 = \emptyset$). If $n+1 \le s+1$ then we proceed and define $\sigma^+ = \delta_{s+1}\restriction n+1$ according to which of the following cases applies. Otherwise we go to step $s+2$.

Before distinguishing the various cases, we first give the following definition:

Definition 5.3: If $\mathcal{A} = \mathcal{A}(A,B,L)$ is any expression involving A, B or L, we write $\mathcal{A}[\sigma,s+1]$ to denote $\mathcal{A}(A[\sigma,s+1], B[\sigma,s+1], L[s+1])$, where

$$A[\sigma,s+1] = (A[s] \cup \bigcup_{\tau\subseteq\sigma} \alpha(\tau,s+1)) - \bigcup_{\tau\subseteq\sigma,\tau\in T^{\mathbb{N}\mathbf{A}}} V(\tau,s+1),$$
$$B[\sigma,s+1] = (B[s] \cup \bigcup_{\tau\subseteq\sigma} \beta(\tau,s+1)) - \bigcup_{\tau\subseteq\sigma,\tau\in T^{\mathbb{N}\mathbf{B}}} V(\tau,s+1).$$

5.2.1 σ is a $\mathbb{P}$-node. Let $\sigma^+ = \sigma^\frown 0$. Go and define $\delta_{s+1}\restriction n+2$, if $n+2 \le s+1$.

5.2.2 σ is a Γ-node. Assume that $\mathcal{R}(\sigma) = (\mathcal{P}_i, x)$, where

$$\mathcal{P}_i: \qquad Z_i = \Phi_i^{A\oplus L} = \Psi_i^{B\oplus L} \Rightarrow Z_i = \Gamma^L_\pi$$

with $\pi \subseteq \sigma$ such that $\mathcal{R}(\pi) = \mathcal{P}_i$. In the following, drop the subscript i and write $\Gamma = \Gamma_\pi$.

1. If $x \in \Phi^{A\oplus L}[\sigma,s+1] \cap \Psi^{B\oplus L}[\sigma,s+1] - \Gamma^L[\sigma,s+1]$, then optimally choose finite sets $\alpha, \beta, \lambda^A, \lambda^B$, according to Definition 5.2, where we take as $P(\langle \alpha,\beta,\lambda^A,\lambda^B\rangle, s+1)$ the relation that holds if and only if

$$\langle x, \alpha\oplus\lambda^A\rangle \in \Phi^s,$$
$$\langle x, \beta\oplus\lambda^B\rangle \in \Psi^s,$$

and

$$\alpha\oplus\lambda^A \subseteq (A\oplus L)[\sigma,s+1],$$
$$\beta\oplus\lambda^B \subseteq (B\oplus L)[\sigma,s+1].$$

Γ-updating. Enumerate $\langle x, \lambda \rangle \in \Gamma^{s+1}$, where

$$\lambda = \lambda^A \cup \lambda^B \cup \bigcup \{\lambda(\rho, s+1) \mid \rho \subseteq \sigma\}.$$

2. We now define the outcome at σ. For this we need to introduce an auxiliary set $G(\sigma, s+1)$ linearly ordered by the (strict) order $<^{\sigma}_{s+1}$ (assume that $G(\tau, 0) = <^{\tau}_0 = \emptyset$, for all τ).

   We first update $G(\sigma, s)$. If there exists a finite set $\lambda \in G(\sigma, s)$ such that $\lambda \not\subseteq L[s+1]$, then let $\hat{\lambda}$ be the $<^{\sigma}_s$-least such set and let

   $$D(\sigma, s+1) = \{\lambda \in G(\sigma, s) \mid \lambda <^s_{\sigma} \hat{\lambda}\};$$

   if no such λ exists, then let $D(\sigma, s+1) = G(\sigma, s)$.

   Let now

   $$G(\sigma, s+1) = D(\sigma, s+1) \cup \{\lambda \mid \langle y, \lambda \rangle \in \Gamma^s \,\&\, \lambda \subseteq L[s+1]\}.$$

   Given any $\lambda \in G(\sigma, s+1)$, let

   $$d(\sigma, \lambda, s+1) = \min\{t \leq s+1 \mid (\forall u)[t \leq u \leq s+1 \Rightarrow \lambda \subseteq L[u]]\}.$$

   Finally, for every $\lambda, \lambda' \in G(\sigma, s+1)$, let $\lambda <^{s+1}_{\sigma} \lambda'$ if and only if

   $$d(\sigma, \lambda, s+1) < d(\sigma, \lambda', s+1) \text{ or } [d(\sigma, \lambda, s+1) = d(\sigma, \lambda', s+1) \,\&\, \lambda < \lambda']$$

   (where we say that $\lambda < \lambda'$ if the canonical index of λ is smaller than the canonical index of λ'). Let $k(\sigma, s+1)$ be the canonical index of $D(\sigma, s+1)$. Define

   $$\sigma^+ = \sigma^\frown k(\sigma, s+1).$$

   Having defined the outcome, next we look for finite sets α, β to be restrained in A, B, respectively, in order to make sure that $x \in \Phi^{A \oplus L} \cap \Psi^{B \oplus L}$, whenever possible.

   For every $\lambda \in D(\sigma, s+1)$, if there exist finite subsets α, β such that

   $$x \in \Phi^{\alpha \oplus \lambda}[s+1] \cap \Psi^{\beta \oplus \lambda}[s+1]$$

   and

   $$\alpha \cap \bigcup \{V(\nu, s+1) \mid \nu \subseteq \sigma \,\&\, \nu \in T^{\mathbf{N^A}}\} = \emptyset,$$

   $$\beta \cap \bigcup \{V(\nu, s+1) \mid \nu \subseteq \sigma \,\&\, \nu \in T^{\mathbf{N^B}}\} = \emptyset,$$

then optimally choose some such finite sets $\alpha(\sigma, \lambda, s+1), \beta(\sigma, \lambda, s+1)$.

If no such finite sets exist, then let $\alpha(\sigma, \lambda, s+1) = \beta(\sigma, \lambda, s+1) = \emptyset$. Let

$$\alpha(\sigma^+, s+1) = \bigcup_{\lambda \in D(\sigma, s+1)} \alpha(\sigma, \lambda, s+1) \quad \text{and}$$

$$\beta(\sigma^+, s+1) = \bigcup_{\lambda \in D(\sigma, s+1)} \beta(\sigma, \lambda, s+1);$$

we will enumerate the elements of $\alpha(\sigma^+, s+1)$ in A^{s+1} and the elements of $\beta(\sigma^+, s+1)$ into B^{s+1}.

Initialization. If $\alpha(\sigma^+, s+1) \neq \alpha(\sigma^+, s)$ or otherwise $\beta(\sigma^+, s+1) \neq \beta(\sigma^+, s)$, then **initialize** all τ such that $\sigma^+ \preceq \tau$, by letting

- $\Gamma_\tau^{s+1} = \emptyset$ and $\Delta_\tau^{s+1} = \emptyset$;
- $c(\tau, z, s+1) =\uparrow$, all z.

In this case move directly to stage $s+2$.

Otherwise, go and define $\delta_{s+1} \upharpoonright n+2$, if $n+2 \leq s+1$.

5.2.3 σ is an $\mathbb{N}^A$-node. Assume that $\mathcal{R}(\sigma) = \mathcal{N}_k^A$, where

$$\mathcal{N}_k^A: \qquad A = \Phi_k^L \Rightarrow \overline{K} = \Delta_\sigma^L.$$

For simplicity, we will omit the subscripts k and σ, thus writing Φ for Φ_k, and Δ for Δ_σ.

In order to measure the length of agreement between $\overline{K}$ and Δ^L, we now introduce the following length of agreement function.

Definition 5.4: Let

$$\ell(\sigma, s+1) = \text{least}\{\langle x, t\rangle \mid x \leq s \,\&\, [t = 0 \,\&\, x \in \overline{K}^{s+1} \,\&\, x \notin \Delta^L[s+1]] \vee (\forall u)[t \leq u \leq s+1 \Rightarrow [x \in \Delta^L[u] \,\&\, x \notin \overline{K}^{s+1}]]\}.$$

If no such $\langle x, t\rangle$ exists, then let $\ell(\sigma, s+1) = s+1$.

Remark 5.5: We notice that if $\{s \mid \sigma \subseteq \delta_s\}$ is infinite and there exist infinitely many stages $s+1$ such that $\ell(\sigma, s+1) = \langle x, t\rangle$, then either $x \in \overline{K} - \Delta^L$ or $x \in \Delta^L - \overline{K}$. Indeed, it is clear that either $x \in \overline{K}$ or $x \in \Delta^L$. If for instance $x \in \overline{K}$, then, under the assumptions, we have that there exist infinitely many stages v such that $x \notin \Delta^L[v]$: a similar argument works if $x \in \Delta^L$.

Let $\sigma^+ = \sigma^\frown \ell(\sigma, s+1)$.

Definition 5.6: We say that $s+1$ is σ-**expansionary** if

$$x \in \overline{K}^{s+1} \,\&\, c(\sigma, x, s) = \uparrow$$

where, say, $\ell(\sigma, s+1) = \langle x, t \rangle$.

We distinguish the following two cases. Let $\ell(\sigma, s+1) = \ell$:

(a) $s+1$ is σ-expansionary.

In this case, define $c(\sigma, x, s+1)$ to be a **new** $c \in \xi_\sigma$.

(b) If $s+1$ is not σ-expansionary, then let $c(\sigma, z, s+1) = \uparrow$, for every z such that $\ell < z$. Define

$$V(\sigma, s+1) = \{c(\sigma, z, s) \mid [z \leq \ell \,\&\, z \notin \overline{K}^{s+1}] \vee z > \ell\} - \bigcup\{\alpha(\rho, s+1) \mid \rho \preceq \sigma\}.$$

Finally, let

$$\alpha(\sigma^+, s+1) = \{c(\sigma, z, s+1) \mid z \leq \ell \;\&\; z \in \overline{K}^{s+1}\}$$

and

$$C(\sigma^+, s+1) = \{z \mid z \leq \ell \;\&\, z \neq x \,\&\; z \in \overline{K}^{s+1}\}.$$

For every $z \in C(\sigma^+, s+1)$ optimally choose (see Definition 5.2) a finite set $\lambda(\sigma, z, s+1)$ such that

$$\langle z, \lambda(\sigma, z, s+1) \rangle \in \Delta^{s+1} \quad \& \quad \lambda(\sigma, z, s+1) \subseteq L[s+1]$$

and let

$$\lambda(\sigma^+, s+1) = \bigcup_{z \in C(\sigma^+, s+1)} \lambda(\sigma, z, s+1).$$

If $x \in \overline{K}^{s+1}$ and $c(\sigma, x, s+1) \in \Phi^L[s+1]$ then optimally choose a finite set λ such that

$$\langle c(\sigma, x, s+1), \lambda \rangle \in \Phi^{s+1} \,\&\, \lambda \subseteq L[s+1]$$

and let $\langle x, \lambda \rangle \in \Delta^{s+1}$.

Initialization. If $\alpha(\sigma^+, s+1) \neq \alpha(\sigma^+, s)$, then initialize all τ such that $\sigma^+ \preceq \tau$. Otherwise, go and define $\delta_{s+1} \upharpoonright n+2$, if $n+2 \leq s+1$.

5.2.4 σ is an $(\mathbb{N}^A, \mathbb{P})$-node. Assume that $\mathcal{R}(\sigma) = (\mathcal{N}, \mathcal{P})$: let $\nu = o(\sigma)$, let $\pi \subseteq \nu$ be such that $\mathcal{R}(\pi) = \mathcal{P}$, and let ℓ be the outcome of ν along σ; finally assume that (omitting obvious subscripts)

$$\mathcal{N}: \qquad A = \Omega^L \Rightarrow \overline{K} = \Delta^L,$$

and

$$\mathcal{P}: \qquad Z = \Phi^{A \oplus L} = \Psi^{B \oplus L} \Rightarrow Z = \Gamma^L.$$

Let also

$$\hat{V}^A_{s+1} = \bigcup_{\hat{\nu} \subset \nu, \hat{\nu} \in T^{\mathbb{N}^A}} V(\hat{\nu}, s+1) \quad \text{and} \quad \hat{V}^B_{s+1} = \bigcup_{\hat{\nu} \subset \nu, \hat{\nu} \in T^{\mathbb{N}^B}} V(\hat{\nu}, s+1).$$

In order to define $E(\sigma, s)$, we need to introduce an auxiliary parameter $H(\sigma, s+1)$ ordered by the (strict) linear order $\prec^s_\sigma$ (assume that $H(\tau, 0) = \prec^0_\tau = \emptyset$, for every τ). The definitions of $E(\tau, t)$ and $H(\tau, t)$ are similar to the definitions of $D(\tau, t)$ and $G(\tau, t)$, respectively, given for the Γ-nodes.

We first update $H(\sigma, s)$. If there exists $x \in H(\sigma, s)$ such that $x \notin \Gamma^L[s+1]$, then let $\hat{x}$ be the $\prec^s_\sigma$-least such number and let

$$E(\sigma, s+1) = \{y \in H(\sigma, s) |\ y \prec^s_\sigma \hat{x}\};$$

if no such x exists, then let $E(\sigma, s+1) = H(\sigma, s)$.

Let now

$$\begin{aligned} H(\sigma, s+1) = E(\sigma, s+1) \cup \{y |\ (\exists \lambda)[\langle y, \lambda\rangle \in \Gamma^s \\ \& \lambda \subseteq L[s+1] \,\&\, y \in \Phi^{(\omega - \hat{V}^A) \oplus \lambda}[s+1] - \Phi^{(\omega - (\hat{V}^A \cup V(\nu))) \oplus \lambda}[s+1]\}. \end{aligned}$$

Given any $y \in H(\sigma, s+1)$, let

$$e(\sigma, y, s+1) = \min\{t \leq s+1 |\ (\forall u)[t \leq u \leq s+1 \Rightarrow y \in \Gamma^L[u]]\}.$$

Finally, for every $y, y' \in H(\sigma, s+1)$, let $y \prec^{s+1}_\sigma y'$ if and only if

$$e(\sigma, y, s+1) < e(\sigma, y', s+1) \quad \text{or} \quad [e(\sigma, y, s+1) = e(\sigma, y', s+1) \,\&\, y < y'].$$

Let $h(\sigma, s+1)$ be the canonical index of $E(\sigma, s+1)$. Define

$$\sigma^+ = \sigma^\frown h(\sigma, s+1).$$

Since $E(\sigma, s+1) \subseteq \Gamma^L[s+1]$, for every $y \in E(\sigma, s+1)$ optimally choose (see Definition 5.2) a finite set $\lambda(\sigma, y, s+1)$ such that

$$\langle y, \lambda(\sigma, y, s+1)\rangle \in \Gamma^s \quad \& \quad \lambda(\sigma, y, s+1) \subseteq L[s+1],$$

and a finite set $\beta(\sigma, y, s+1)$ such that $\beta(\sigma, y, s+1) \cap \hat{V}^B_{s+1} = \emptyset$ and

$$y \in \Psi^{\beta(\sigma,y,s+1)\oplus\lambda(\sigma,y,s+1)}$$

(if no such finite set exists then simply let $\beta(\sigma, y, s+1) = \emptyset$). Let

$$\beta(\sigma^+, s+1) = \bigcup_{y \in E(\sigma,s+1)} \beta(\sigma, y, s+1).$$

Finally, let

$$\lambda(\sigma^+, s+1) = \bigcup_{y \in E(\sigma,s+1)} \lambda(\sigma, y, s+1).$$

Initialization. If $\beta(\sigma^+, s+1) \neq \beta(\sigma^+, s)$, then initialize all τ such that $\sigma^+ \preceq \tau$. Otherwise, go and define $\delta_{s+1} \upharpoonright n+2$, if $n+2 \leq s+1$.

5.2.5 σ is an $\mathbb{N}^B$-node. Assume that $\mathcal{R}(\sigma) = \mathcal{N}^B_k$. This case is similar to the case of an $\mathbb{N}^A$-node, but interchanging A with B, while considering the requirement

$$\mathcal{N}^B_k: \qquad B = \Phi^L_k \Rightarrow \overline{K} = \Delta^L_\sigma.$$

5.2.6 σ is an $(\mathbb{N}^B, \mathbb{P})$-node. This case is similar to the case of an $(\mathbb{N}^A, \mathbb{P})$-node, but interchanging A with B and Φ with Ψ, while considering the requirements (assuming $\mathcal{R}(\sigma) = (\mathcal{N}^B_k, \mathcal{P}_i)$)

$$\mathcal{N}^B_k: \qquad B = \Phi^L_k \Rightarrow \overline{K} = \Delta^L_\sigma,$$

and

$$\mathcal{P}_i: \qquad Z_i = \Phi_i^{A\oplus L} = \Psi_i^{B\oplus L} \Rightarrow Z_i = \Gamma^L_\sigma.$$

Notice also that in this case we define finite sets $\alpha(\sigma, y, s+1)$ (instead of $\beta(\sigma, y, s+1)$) and $\alpha(\sigma^+, s+1)$ (instead of $\beta(\sigma^+, s+1)$).

5.2.7 Final updating. At the end of stage $s+1$ let

$$A^{s+1} = (A[s] \cup \bigcup_{\tau\subseteq\sigma} \alpha(\tau, s+1)) - \bigcup_{\tau\subseteq\sigma, \tau\in T^{\mathbb{N}^A}} V(\tau, s+1)$$

and

$$B^{s+1} = (B[s] \cup \bigcup_{\tau\subseteq\sigma} \beta(\tau, s+1)) - \bigcup_{\tau\subseteq\sigma, \tau\in T^{\mathbb{N}^B}} V(\tau, s+1).$$

For every $\sigma \subseteq \delta_{s+1}$, let

$$\begin{aligned} \Gamma^{s+1}_\sigma &= \Gamma^s_\sigma \cup \{\langle x, \lambda\rangle \mid \langle x, \lambda\rangle \text{ enumerated into } \Gamma^{s+1}_\sigma\}, \\ \Delta^{s+1}_\sigma &= \Delta^s_\sigma \cup \{\langle x, \lambda\rangle \mid \langle x, \lambda\rangle \text{ enumerated into } \Delta^{s+1}_\sigma\}. \end{aligned}$$

6. The verification

We first show

LEMMA 6.1: *For every n,*

(1) $\sigma_n = \liminf_s \delta_s \upharpoonright n$ *exists.*

(2) σ_n *is eventually never initialized.*

(3) $\lim_s \alpha(\sigma_n, s)$, $\lim_s \beta(\sigma_n, s)$, *and* $\lim_s \lambda(\sigma_n, s)$ *exist and are finite; moreover, writing* $\lambda(\sigma_n) = \lim_s \lambda(\sigma_n, s)$, *we have that* $\lambda(\sigma_n) \subseteq L$.

(4) *If* $\tau \subset \sigma_n$ *is a* Γ*-node, with* $\mathcal{R}(\tau) = (\mathcal{P}, x)$, *and* $\pi \subseteq \tau$ *is such that* $\mathcal{R}(\pi) = \mathcal{P}$, *and* k *is the outcome at* τ *along* σ_n, *then*

$$D_k = \{\lambda |\ \langle x, \lambda\rangle \in \Gamma_\pi^L \ \&\ \lambda \subseteq L]\},$$

and for almost all s, *if* $\tau^\frown k' \subseteq \delta_s$, *then* $D_k \subseteq D_{k'}$.

(5) *If* $\tau \subset \sigma_n$ *is an* $(\mathbb{N}, \mathbb{P})$*-node, with* $\mathcal{R}(\tau) = (\mathcal{N}, \mathcal{P})$, *and* $\pi \subseteq \tau$ *is such that* $\mathcal{R}(\pi) = \mathcal{P}$, *and* h *is the outcome at* τ *along* σ_n, *then*

$$D_h = \{x |\ (\exists s)[x \in H(\tau, s) \ \&\ x \in \Gamma_\pi^L]\},$$

and for almost all s, *if* $\tau^\frown h' \subseteq \delta_s$, *then* $D_h \subseteq D_{h'}$.

Proof: The proof is by induction on n. For $n = 0$ the claim is trivial, being $\sigma_0 = \emptyset$.

Suppose now that the claim is true of n. Let $\sigma_n = \liminf_s \delta_s \upharpoonright n$, and for every $\tau \preceq \sigma_n$ let $\alpha(\tau) = \lim_s \alpha(\tau, s)$, $\beta(\tau) = \lim_s \beta(\tau, s)$, $\lambda(\tau) = \lim_s \lambda(\tau, s)$.

Moreover,

Definition 6.2: Let t_{σ_n} be the least stage such that, for every $s \geq t_{\sigma_n}$,

- for all $\tau \prec_L \sigma_n$, $\tau \not\subseteq \delta_s$;
- σ_n is not initialized at s;
- for all $\tau \preceq \sigma_n$

$$\alpha(\tau, s) = \alpha(\tau) \quad \beta(\tau, s) = \beta(\tau) \quad \lambda(\tau, s) = \lambda(\tau);$$

- for every Γ-node $\tau \subset \sigma_n$, if k is the outcome at τ along σ_n, then for every k',

$$\tau^\frown k' \subseteq \delta_s \Rightarrow D_k \subseteq D_{k'};$$

- for every $(\mathbb{N}, \mathbb{P})$-node $\tau \subset \sigma_n$, if h is the outcome at τ along σ_n, then for every h',
$$\tau^\frown h' \subseteq \delta_s \Rightarrow D_h \subseteq D_{h'}.$$

We distinguish the following cases, according as σ_n is a $\mathbb{P}$-node, a Γ-node, an $\mathbb{N}$-node, or an $(\mathbb{N}, \mathbb{P})$-node.

CASE 1: σ_n is a $\mathbb{P}$-node. Then obviously $\sigma_{n+1} = \liminf_s \delta_s \restriction n+1 = \sigma_n{}^\frown 0$. The other conditions are trivially checked.

CASE 2: σ_n is a Γ-node. Assume that $\mathcal{R}(\sigma_n) = (\mathcal{P}_i, x)$, and let $\pi \subseteq \sigma_n$ be such that $\mathcal{R}(\pi) = \mathcal{P}_i$. In order to prove (4), let $D = \{\lambda |\ \langle x, \lambda \rangle \in \Gamma_\pi \,\&\, \lambda \subseteq L\}$. Then

CLAIM: *The set D is finite, and $\sigma_{n+1} = \sigma_n{}^\frown u$, where u is the canonical index of D.*

To prove the claim, first notice that D contains only finite sets λ, such that we enumerate an axiom $\langle x, \lambda \rangle \in \Gamma^s_\pi$, while acting, at some stage s, at some Γ-node $\pi' \supseteq \pi$, with $\mathcal{R}(\pi') = (\mathcal{P}_i, x)$.

Consider *only* axioms of this form enumerated at stages $s \geq t_{\sigma_n}$.

SUBLEMMA 1: *If $\sigma_n \prec_L \pi'$ then $\lambda \not\subseteq L$.*

Proof: If $\sigma_n \prec_L \pi'$, then there exists some longest τ such that $\pi \subseteq \tau \subset \sigma_n$, and the outcome o at τ along π' is such that $\sigma_n \prec_L \tau^\frown o$. We have the following possibilities:

(a) τ is a Γ-node, where, say, $\mathcal{R}(\tau) = (\mathcal{R}(\pi''), y)$. Let k be the outcome at τ along σ_n; then there exists k' such that $k < k'$ and $\tau^\frown k' \subseteq \pi'$. Then by induction
$$D_k = \{\lambda |\ \langle x, \lambda \rangle \in \Gamma_{\pi''} \,\&\, \lambda \subseteq L\}.$$
Suppose that $s \geq t_{\sigma_n}$ is a stage such that $\tau^\frown k' \subseteq \delta_s$. Then by induction, there exist a finite set $\hat{\lambda} \in D_{k'} - D_k$, and, thus, $\hat{\lambda} \not\subseteq L$. On the other hand, if $\langle x, \lambda \rangle \in \Gamma_\pi$ is an axiom we define at s, while acting at π', then we have that $\lambda \supseteq \hat{\lambda}$. Hence $\lambda \not\subseteq L$.

(b) τ is an $\mathbb{N}$-node; assume for definiteness that τ is an $\mathbb{N}^A$-node: similar arguments apply for $\mathbb{N}^B$-nodes. Let $\ell = \langle y, u \rangle$ be the outcome at τ along σ_n. Thus there exists ℓ' such that $\ell < \ell'$ and $\tau^\frown \ell' \subseteq \pi' \subseteq \delta_s$. By definition of t_{σ_n} and since we are assuming to take action at a stage $s \geq t_{\sigma_n}$, we conclude that ℓ is not finitary, i.e. $y \notin \Delta^L_\tau$, but, at stage s, we have that $y \in \Delta^L_\tau[s]$, and $y \in C(\tau^\frown \ell', s)$; thus if $\langle x, \lambda \rangle \in \Gamma^s_\pi$ is the axiom we define at s at π', then we have that $\lambda(\tau, y, s) \subseteq \lambda$. Since $\lambda(\tau, y, s) \not\subseteq L$, we have that $\lambda \not\subseteq L$.

(c) τ is an $(\mathbb{N}, \mathbb{P})$-node; assume for definiteness that τ is an $(\mathbb{N}^A, \mathbb{P})$-node: similar arguments apply for $(\mathbb{N}^B, \mathbb{P})$-nodes. Let h be the outcome of τ along σ_n, and let $s \geq t_{\sigma_n}$ be such that $\pi' \subseteq \delta_s$. Thus there exists h', with $h < h'$ such that $\tau^\frown h' \subseteq \pi' \subseteq \delta_s$. It follows by induction that

$$D_h = \{x | \, (\exists t)[x \in H(\tau, t) \,\&\, x \in \Gamma^L_{\pi''}]\},$$

where $\mathcal{R}(\tau) = (\mathcal{R}(\nu'), \mathcal{R}(\pi''))$, for some $\mathbb{N}$-node ν' and $\mathbb{P}$-node π''. Since $\liminf_s \delta_s \upharpoonright |\tau| + 1 = \tau^\frown h$, it follows by induction that there must exist $x \in D_{h'}$ and a finite set $\lambda(\tau, x, s)$ such that $x \notin \Gamma^L_{\pi''}$, and $x \in \Gamma^{\lambda(\tau,x,s)}_{\pi''}$ and $\lambda(\tau, x, s) \subseteq L[s]$. The construction ensures that if $\langle x, \lambda\rangle \in \Gamma^s_\pi$ is the axiom we define at s at π', then $\lambda(\tau, x, s) \subseteq \lambda$: but $\lambda(\tau, x, s) \not\subseteq L$, therefore $\lambda \not\subseteq L$. ∎

We have thus shown that the set D, where

$$D = \{\lambda | \, \langle x, \lambda\rangle \in \Gamma_\pi \,\&\, \lambda \subseteq L\}\},$$

is finite, since this set can contain only finite sets λ such that an axiom $\langle x, \lambda\rangle \in \Gamma_\pi$ has been enumerated only while acting at σ_n, or at some stage $s < t_{\sigma_n}$: on the other hand we either define at some stage $s \geq t_{\sigma_n}$ at σ_n some axiom $\langle x, \lambda\rangle \in \Gamma_\pi$ with $\lambda \subseteq L$, in which case we eventually stop appointing axioms at σ_n; or otherwise $D = \emptyset$.

It is now easy to see that $\sigma_{n+1} = \sigma_n{}^\frown u$ where u is the canonical index of D, and that there exists a stage t such that, for every $s \geq t$, if $\sigma_n{}^\frown u' \subseteq \delta_s$ then $D \subseteq D_{u'}$.

Finally, we show (3) for σ_{n+1}. Given any $\lambda \in D$, since we always appoint new coding markers to enter $V(\nu, s)$, for $\nu \subseteq \sigma_n$, either at some stage $s \geq t_{\sigma_n}$ we find some finite sets α and β such that $x \in \Phi^{\alpha\oplus\lambda}[s+1] \cap \Psi^{\beta\oplus\lambda}[s+1]$ and

$$\alpha \cap \bigcup\{V(\nu, s+1) | \, \nu \subseteq \sigma \,\&\, \nu \in T^{\mathbb{N}^A}\} = \emptyset,$$
$$\beta \cap \bigcup\{V(\nu, s+1) | \, \nu \subseteq \sigma \,\&\, \nu \in T^{\mathbb{N}^B}\} = \emptyset,$$

and, in this case, $\alpha(\sigma_n, \lambda) = \lim_s \alpha(\sigma_n, \lambda, s)$ equals some such α, and $\beta(\sigma_n, \lambda) = \lim_s \beta(\sigma_{n+1}, \lambda, s)$ equals some such β; or for every $s \geq t_{\sigma_n}$,

$$\alpha(\sigma_n, \lambda, s) = \beta(\sigma_n, \lambda, s) = \emptyset.$$

In any case $\lim_s \alpha(\sigma_{n+1}, s)$ and $\lim_s \beta(\sigma_{n+1}, s)$ exist. Hence, σ_{n+1} eventually does not initialize any string τ, with $\sigma_{n+1} \preceq \tau$.

CASE 3: σ_n is an $\mathbb{N}^A$-node. Assume that $\mathcal{R}(\sigma_n) = \mathcal{N}_k$. Let

$$\Delta_{\sigma_n} = \bigcup_{s \geq t_{\sigma_n}} \Delta^s_{\sigma_n}.$$

Since $\overline{K} \not\leq_e L$, and, thus, $\overline{K} \neq \Delta^L_{\sigma_n}$, it follows from the observations in Remark 5.5, that $\ell = \liminf_s \ell(\sigma_n, s)$ is finite. Thus $\sigma_{n+1} = \sigma_n{}^\frown\ell$.

Clearly $\lim_s C(\sigma_{n+1}, s) = C(\sigma_{n+1})$ exists and is finite, being

$$C(\sigma_{n+1}) = \{z |\ z \leq \ell \;\&\; z \in \overline{K}\}.$$

On the other hand, it is easy to see that $\lim_s c(\sigma_n, z, s) = c(\sigma_n, z)$ exists for all $z \in C(\sigma_{n+1})$, hence $\lim_s \alpha(\sigma_{n+1}, s) = \{c(\sigma_n, z) |\ z \in C(\sigma_{n+1})\}$. Thus σ_{n+1} eventually stops initializing lower priority strings τ.

Moreover, for every $x \in C(\sigma_{n+1})$, we are eventually able to appoint some (optimally chosen) finite set $\lambda(\sigma_n, x)$ such that $\lambda(\sigma_n, x) = \lim_s \lambda(\sigma_n, x, s)$, with $x \in \Delta^{\lambda(\sigma_n, x)}_{\sigma_n}$ and $\lambda(\sigma_n, x) \subseteq L$. Therefore $\lambda(\sigma_{n+1}) = \lim_s \lambda(\sigma_n, s)$ exists and is finite, being $\lambda(\sigma_{n+1}) = \bigcup_{x \in C(\sigma_{n+1})} \lambda(\sigma_n, x)$. We have also shown that $\lambda(\sigma_{n+1}) \subseteq L$.

CASE 4: σ_n is an $(\mathbb{N}^A, \mathbb{P})$-node. Assume that $\mathcal{R}(\sigma_n) = (\mathcal{N}, \mathcal{P})$; let $\pi \subseteq \sigma_n$ be such that $\mathcal{R}(\pi) = \mathcal{P}$, and let $\nu = o(\sigma_n)$.

We first show

SUBLEMMA 2: *$\liminf_s h(\sigma_n, s)$ is finite. In fact $\liminf h(\sigma_n, s) = h$, where*

$$D_h = \{x |\ (\exists s)[x \in H(\sigma_n, s) \,\&\, x \in \Gamma^L_\pi]\}.$$

Proof: In order to show that (5) is true, we first observe:

CLAIM: *The set E, where*

$$E = \{x |\ (\exists s)[x \in H(\sigma_n, s) \,\&\, x \in \Gamma^L_\pi]\},$$

is finite.

Indeed, clearly E contains only numbers x, such that we enumerate an axiom $\langle x, \lambda \rangle \in \Gamma^s_\pi$, while acting, at some stage s, at some Γ-node $\pi' \supseteq \pi$, with $\mathcal{R}(\pi') = (\mathcal{P}, x)$.

Consider now all Γ-node $\pi' \supseteq \pi$, with $\mathcal{R}(\pi') = (\mathcal{P}, x)$, for which we define axioms $\langle x, \lambda \rangle \in \Gamma^s_\pi$ *only* at stages $s \geq t_{\sigma_n}$.

We distinguish the following two subcases.

SUBCASE 1: $\sigma_n \prec_L \pi'$. We argue in this case as in the proof of Sublemma 1 that, for every axiom $\langle x, \lambda\rangle \in \Gamma_\pi$ which we have enumerated at a stage $s \geq t_{\sigma_n}$ at π', we have $\lambda \not\subseteq L$.

SUBCASE 2: $\sigma_n \subset \pi'$. Given any stage t, let us say that $x \in H(\sigma_n, t)$ *because of* π', if there is an axiom $\langle x, \lambda\rangle \in \Gamma_\pi$ appointed at π', at some stage $s \leq t$, such that, letting

$$\hat{V}_t^A = \bigcup_{\hat{\nu}\subset\nu,\hat{\nu}\in T^{\mathbb{N}^A}} V(\hat{\nu}, t),$$

we have that $\lambda \subseteq L[t]$ and $x \in \Phi_\pi^{(\omega-\hat{V}^A)\oplus\lambda}[t] - \Phi_\pi^{(\omega-(\hat{V}^A\cup V(\nu))\oplus\lambda}[t]$.

We claim that there is no t such that $x \in H(\sigma_n, t)$, because of π': assume for a contradiction otherwise, and let t be such a stage. Since we consider only axioms appointed at π' only at stages $s \geq t_{\sigma_n}$, we may assume that $t \geq t_{\sigma_n}$. Thus, there must exist an axiom $\langle x, \lambda\rangle \in \Gamma_\pi$, appointed at some stage s such that $t_{\sigma_n} \leq s \leq t$, while acting at π'; hence $x \in \Phi_\pi^{A\oplus L}[s]$, and thus there exists a finite set α such that $x \in \Phi_\pi^{\alpha\oplus\lambda}$ and $\alpha \oplus \lambda \subseteq A \oplus L[s]$. Since $\sigma_n = \liminf_s \delta_s \restriction n$, and by our choice of t_{σ_n}, it easily follows that

$$(\hat{V}_t^A \cup V(\nu, t)) \cap \alpha = \emptyset$$

since at stages $u \geq s$ only new numbers (thus numbers not in α) can be appointed as new ν'-coding markers (with $\nu' \subseteq \nu$ and $\nu' \in T^{\mathbb{N}^A}$) and possibly enter V_t^A, or $V(\nu, t)$. Since this holds of every possible axiom appointed at π' at any s such that $t_{\sigma_n} \leq s \leq t$, we have a contradiction. We therefore conclude that the case $x \in H(\sigma_n, t)$ because of π' does not hold. Hence, no $\pi' \supseteq \sigma_n$ can contribute elements into $H(\sigma_n, t)$ for any $t \geq t_{\sigma_n}$.

We have thus shown that the set E, where

$$E = \{x| \ (\exists s)[x \in H(\sigma_n, s) \ \& \ x \in \Gamma_\pi^L]\},$$

is finite, since this set can contain only numbers x such that either

- axioms $\langle x, \lambda\rangle \in \Gamma_\pi^s$ are appointed at stages $s < t_{\sigma_n}$, or
- axioms $\langle x, \lambda\rangle \in \Gamma_\pi^s$ are appointed at Γ-nodes π' (with $\mathcal{R}(\pi') = (\mathcal{P}, x)$) such that $\pi \subseteq \pi' \subseteq \sigma_n$, but only for finitely many numbers x does an axiom $\langle x, \lambda\rangle \in \Gamma_\pi$ get appointed at any of these nodes.

Let now h be the canonical index of E. We are now in a position to show that

$$\sigma_{n+1} = \liminf_s \delta_s \restriction n + 1 = \sigma_n{}^\frown h.$$

Clearly there exists a stage $s_0 \geq t_{\sigma_n}$ such that $D_h \subseteq D_{h(\sigma_n,s)}$, for every $s \geq s_0$, and

$$(\forall s \geq s_0)(\forall y \in D_h)(\forall x \in H(\sigma, s) - D_h)[y \preceq^s_{\sigma_n} x]$$

and $\preceq_{\sigma_n} = \lim_s \preceq^s_{\sigma_n}$ exists on D_h, where, for any $y, y' \in D_h$, we have $y \prec_{\sigma_n} y'$ if and only if, for some t_0, t_1 with $t_0 \leq t_1$,

$$(\forall s \geq t_0)[y \in \Gamma^L_\pi[s]] \,\&\, y' \notin \Gamma^L_\pi[t_1].$$

To show that there are infinitely many stages s such that $h(\sigma_n, s) = h$, we show that for every $t \geq s_0$ there exists $s > t$ such that $h(\sigma_n, s) = h$. To this end, let $t \geq s_0$. Suppose that $s' \geq t$ is such that $\sigma_n \subseteq \delta_{s'}$: then $D_h \subseteq D_{h(\sigma_n,s')}$. Let us assume that $x \in D_{h(\sigma_n,s')} - D_h$, and x is the $\prec^{s'}_{\sigma_n}$-least such element; clearly $x \notin \Gamma^L_\pi$, and, for every $y \in D_h$, $y \prec^{s'}_{\sigma_n} x$. It follows that at the least stage $s > s'$ such that $\sigma_n \subseteq \delta_s$ and $x \notin \Gamma^L_\pi[s]$, we define $E(\sigma_n, s) = D_h$, hence $h(\sigma_n, s) = h$. ∎

It follows that we eventually appoint some optimally chosen finite sets

$$\beta(\sigma_n, y) = \lim_s \beta(\sigma_n, y, s), \quad \lambda(\sigma_n, y) = \lim_s \lambda(\sigma_n, y, s)$$

for every $y \in D_h$, such that $y \in \Psi^{\beta(\sigma_n,y)\oplus\lambda(\sigma_n,y)}$, and $\lambda(\sigma_n, y) \subseteq L$, and thus the set $\beta(\sigma_{n+1}) = \lim_s \beta(\sigma_{n+1}, s)$ exists and is finite, being $\beta(\sigma_{n+1}) = \bigcup_{y\in D_h} \beta(\sigma_n, y)$. Finally we observe that $\lim_s \lambda(\sigma_{n+1}, s) = \lambda(\sigma_{n+1})$, where

$$\lambda(\sigma_{n+1}) = \bigcup_{y\in D_h} \lambda(\sigma_n, y),$$

and $\lambda(\sigma_{n+1}) \subseteq L$. Thus (3) is true of σ_{n+1}.

It then also follows that σ_{n+1} eventually stops initializing lower priority strings τ.

CASE 5: σ_n is an $\mathbb{N}^B$-node. The verification is similar to Case 3, but interchanging A with B.

CASE 6: σ_n is an $(\mathbb{N}^B, \mathbb{P})$-node. The verification is similar to Case 4, but interchanging A with B, and Φ with Ψ. ∎

Definition 6.3: By Lemma 6.1, let f be the infinite path through T such that, for every n, $f \restriction n = \sigma_n$. The path f is called the **true path**.

LEMMA 6.4: *For every k, the requirements $\mathcal{N}_k^A$ and $\mathcal{N}_k^B$ are satisfied.*

Proof: Assume that n is such that $\mathcal{R}(f\restriction n) = \mathcal{N}_k^A$ (a similar argument applies if $f\restriction n$ is an $\mathbb{N}^B$-node). Then by Lemma 6.1, Case 3, $\liminf_s \ell(f\restriction n, s)$ exists.

The claim easily follows from the following sublemma.

SUBLEMMA 3: *If $A = \Phi_k^L$ then $\overline{K} = \Delta_{\sigma_n}^L$ (where Δ_{σ_n} is defined as in Case 3 of the proof of Lemma 6.1).*

Proof: Let

$$X = \{c|\ (\exists z)(\exists s \geq t_{\sigma_n})[c = c(\sigma_n, z, s)\}.$$

We in fact show that $A(c) \neq \Phi_k^L(c)$ for some $c \in X$. Assume the contrary. Let $\lim_s \ell(\sigma_n, s) = \langle x, u\rangle$, thus $\overline{K}(x) \neq \Delta_{\sigma_n}(x)$.

Assume first that $x \in \overline{K}$. By definition of the length of agreement function and by construction, we have that $c = \lim_s c(\sigma_n, x, s)$ exists, and $c \in A$. Since we are assuming that $A(c) = \Phi_k^L(c)$, we are eventually able to appoint an axiom $\langle x, \lambda\rangle \in \Delta_{\sigma_n}$ with $c \in \Phi_k^\lambda$ and $\lambda \subseteq L$, thus giving $x \in \Delta_{\sigma_n}^L$, contradiction.

Assume now that $x \notin \overline{K}$. Then for every $s \geq t_{\sigma_n}$, it follows by construction that $c(\sigma_n, x, s) \notin A$. Since

$$(\forall\lambda)[\langle x, \lambda\rangle \in \Delta_{\sigma_n} \Rightarrow (\exists s \geq t_{\sigma_n})[\langle c(\sigma_n, x, s), \lambda\rangle \in \Phi_k],$$

it follows that $x \notin \Delta_{\sigma_n}^L$. ∎

LEMMA 6.5: *For every i the requirement $\mathcal{P}_i$ is satisfied.*

Proof: Given i, we want to show that

$$Z_i = \Phi_i^{A\oplus L} = \Psi_i^{B\oplus L} \Rightarrow Z_i =^* \Gamma^L$$

where $\Gamma = \Gamma_\sigma$ is the e-operator that we construct at nodes $\tau \supseteq \sigma$, with $\sigma \subset f$ such that $\mathcal{R}(\sigma) = \mathcal{P}$.

For simplicity, throughout the following proof, we will omit the subscript i. Let x be given. Let $\tau \subset f$ be the Γ-node, such that $\mathcal{R}(\tau) = (\mathcal{P}, x)$, and, by Definition 6.2 and Lemma 6.1, let t_τ be a stage such that for every $s \geq t_\tau$,

- for all $\tau' \prec_L \tau$, $\tau' \not\subseteq \delta_s$;
- for all $\rho \subseteq \tau$, $\lambda(\rho, s) = \lambda(\rho, t_\tau)(= \lambda(\rho))$ and $\lambda(\rho) \subseteq L$;
- for every node $\nu \subseteq \tau$, if ν^- is a $\mathbb{N}^A$-node then $\alpha(\nu, s) = \alpha(\nu, t_\tau)$, or $\beta(\nu, s) = \beta(\nu, t_\tau)$ if ν^- is a $\mathbb{N}^B$-node.

First assume that $x \in \Phi^{A\oplus L} \cap \Psi^{B\oplus L}$. Then there exists a stage t such that, for every $s \geq t$, $x \in \Phi^{A\oplus L}[s] \cap \Psi^{B\oplus L}[s]$. Since $\{s|\ \tau \subseteq \delta_s\}$ is infinite, we can eventually find a stage $t_0 \geq t_\tau$ such that at t_0 we appoint finite sets $\alpha, \beta, \lambda^A, \lambda^B$ such that

$$\alpha \oplus \lambda^A \subseteq A \oplus L, \qquad \beta \oplus \lambda^B \subseteq B \oplus L,$$

and $x \in \Phi^{\alpha\oplus\lambda^A} \cap \Psi^{\beta\oplus\lambda^B}$ and, for every $s \geq t_0$,

$$\alpha \cap \bigcup_{\nu\subseteq\tau, \nu\in T^{\mathbb{N}^A}} V(\nu, s) = \emptyset \quad \text{and} \quad \beta \cap \bigcup_{\nu\subseteq\tau, \nu\in T^{\mathbb{N}^B}} V(\nu, s) = \emptyset,$$

and we appoint an axiom

$$\langle x, \lambda^A \cup \lambda^B \cup \bigcup\{\lambda(\rho)|\ \rho \subseteq \tau\}\rangle \in \Gamma.$$

Therefore $x \in \Gamma^L$, since by Lemma 6.1 $\lambda(\rho) \subseteq L$, for every $\rho \subseteq \tau$.

Assume now that $Z = \Phi^{A\oplus L} = \Psi^{B\oplus L}$, and let, for a contradiction, $x \in \Gamma^L - Z$. Assume further that axioms of the form $\langle x, \lambda\rangle \in \Gamma$ are only enumerated at stages $s \geq t_\sigma$, where t_σ is as given in Definition 6.2.

Suppose that $s_0 \geq t_\sigma$ is the least stage such that we appoint at some $\pi' \supseteq \sigma$ finite sets $\alpha = \alpha(\pi', s_0), \beta = \beta(\pi', s_0), \lambda^A(\pi', s_0), \lambda^B(\pi', s_0)$ and we enumerate an axiom $\langle x, \lambda\rangle \in \Gamma$ with $\lambda^A(\pi', s_0) \cup \lambda^B(\pi', s_0) \subseteq \lambda \subseteq L$.

Then there must be some $\mathbb{N}$-nodes $\nu \subset f$, such that our extracting activity on behalf of the $\mathcal{N}$-requirements located at those $\mathbb{N}$-nodes prevents us from reinstating $x \in \Phi_i^{A\oplus\lambda} \cap \Psi_i^{B\oplus\lambda}$ while acting at τ (where $\tau \subset f$ is the Γ-node, such that $\mathcal{R}(\tau) = (\mathcal{P}, x)$), via enumeration or re–enumeration of suitable finite sets in A or B.

Let ν be the least $\mathbb{N}$-node (assume for definiteness that ν is an $\mathbb{N}^A$-node: similar arguments apply if ν is an $\mathbb{N}^B$-node) such that $\sigma \subseteq \nu \subseteq \tau$ and

$$x \in \Phi^{(\omega-\hat{V}^A)\oplus\lambda} - \Phi^{(\omega-(\hat{V}^A\cup V(\nu)))\oplus\lambda},$$

where

$$\hat{V}^A = \{y|\ (\exists^\infty s)(\exists\hat{\nu} \subset \nu[\hat{\nu} \in T^{\mathbb{N}^A} \,\&\, y \in V(\hat{\nu}, s)]\}$$

and

$$V(\nu) = \{y|\ (\exists^\infty s)[y \in V(\nu, s)]\}.$$

Let $\rho \subset f$ be the $(\mathbb{N}, \mathbb{P})$-node immediately following ν on the true path, with $\mathcal{R}(\rho) = (\mathcal{R}(\nu), \mathcal{P})$. It follows from Lemma 6.1(5) that $x \in D_h$, where h is the outcome at ρ along f. By minimality, there is no $\mathbb{N}^B$-node ν' such that $\sigma \subseteq \nu' \subseteq \nu$ and the extracting activity demanded by $\mathcal{R}(\nu')$ interferes with restraining some finite set $\beta \subseteq B$ to get $x \in \Psi^{B\oplus\lambda}$. Therefore, we can eventually restrain a finite set $\beta(\pi, x) = \lim_s \beta(\pi, x, s) \subseteq B$ such that $x \in \Psi^{\beta(\pi,x)\oplus\lambda}$.

This shows that $x \in \Psi^{B\oplus L} - \Phi^{A\oplus L}$, contradicting the hypothesis that $\Phi^{A\oplus L} = \Psi^{B\oplus L}$.

This concludes the proof of the theorem. ∎

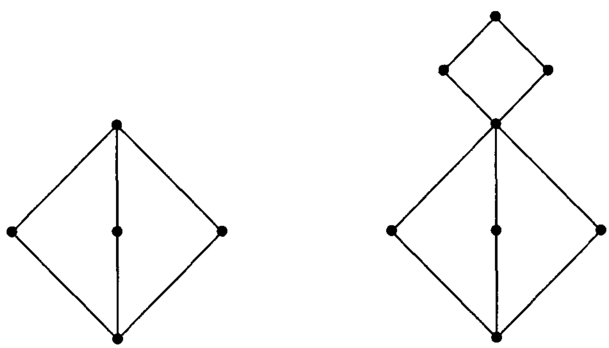
7. Lattice embeddings

[Lac72] shows that the nondistributive lattice M_3 (see Figure 1) can be embedded into the low c.e. Turing degrees. By [MC85], it follows that every lattice which is known to be embeddable into the low c.e. Turing degrees (and thus M_3 as well) can be embedded into $\mathfrak{S}$: since, under such an embedding, the e-degree corresponding to the top element is incomplete and, thus, branching by Theorem 1.3, there follows*:

THEOREM 7.1: *The lattice S_8 of Figure 1 can be embedded into $\mathfrak{S}$.*

Proof: Trivial.

Since S_8 is not embeddable into the c.e. Turing degrees (see [LS80]), it follows that the class of finite lattices that are embeddable into $\mathfrak{S}$ properly extends the class of finite lattices that are known to be embeddable into the c.e. Turing degrees.



M_5 S_8

Figure 1.

* We thank R. Shore for pointing out to us this consequence of Theorem 1.3.

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